

ON A LINEAR PROBLEM ARISING IN DYNAMIC BOUNDARIES

MARCELO M. DISCONZI

ABSTRACT. We study a linear problem that arises in the study of dynamic boundaries, in particular in free boundary problems in connection with fluid dynamics. The equations are also very natural and of interest on their own.

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1. INTRODUCTION.

Consider the problem

$$\begin{cases} \Delta f = 0 & \text{in } \Omega, \\ \ddot{f} - \kappa \bar{\Delta} \partial_\nu f = G & \text{on } \partial\Omega, \\ f(0, \cdot) = f_0, \dot{f}(0, \cdot) = f_1 & \text{on } \partial\Omega, \end{cases} \quad \begin{array}{l} (1.1a) \\ (1.1b) \\ (1.1c) \end{array}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain whose boundary $\partial\Omega$ is an $n-1$ -dimensional manifold embedded in \mathbb{R}^n ; $\bar{\Delta}$ is the Laplacian on $\partial\Omega$ and Δ the Laplacian in \mathbb{R}^n ; ∂_ν is the outer normal derivative on $\partial\Omega$; κ is a positive constant; $f : [0, T] \times \Omega \rightarrow \mathbb{R}$ is the unknown, $T > 0$; $G : [0, T] \times \partial\Omega \rightarrow \mathbb{R}$, $f_0 : \partial\Omega \rightarrow \mathbb{R}$, and $f_1 : \partial\Omega \rightarrow \mathbb{R}$ are given functions; and “ $\dot{\cdot}$ ” means derivative with respect to t , where we write $f = f(t, x)$, $t \in [0, T]$, $x \in \bar{\Omega}$. We shall elaborate an appropriate notion of weak solution to (1.1), then establish the existence and uniqueness of weak solutions on any time interval $[0, T]$.

Let us discuss some motivations to study (1.1). In this regard, it is perhaps worthwhile to start noticing that, from a PDE perspective, problem (1.1) is very natural. Without the normal derivative ∂_ν on the second term on the left-hand side of (1.1b), the problem decouples: (1.1b)-(1.1c) becomes a wave equation on the boundary, which can be solved by standard techniques, and equation (1.1a) says that this solution on $\partial\Omega$ is extended to the interior via the unique harmonic extension of $f|_{\partial\Omega}$. A similar procedure is no longer possible when the term ∂_ν is present, as in (1.1). The introduction of the normal derivative can be viewed as one of the simplest ways of modifying the wave equation on the boundary as to make it dependent on the interior values of f .

A more direct motivation to investigate (1.1) is that it arises at the linearized level in the study of the (incompressible) free boundary Euler equations, as we now explain.

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Consider the motion of an inviscid incompressible fluid within a bounded region of space, and suppose further that the boundary of the region confining the fluid is not rigid, being allowed to move according to the pressure exerted by the fluid (hence the name “free boundary”). This is the situation, for example, in a liquid drop, or in Newtonian self-gravitating fluid bodies, such as stars [40, 42, 45, 61] (the analogue problem for viscous fluids was first and extensively studied by Solonnikov [43, 53, 54, 55, 55, 57, 58, 59], with some more recent advances found in [36, 47, 49, 50, 51] and references therein). In such situations, the domain containing the fluid changes over time. One thus writes $\Omega(t)$, and $\Omega(t)$ becomes one of the unknowns of the problem.

The equations of motion describing the situation of the previous paragraph are the well-known free boundary Euler equations:

$$\begin{cases} \frac{\partial u}{\partial t} + \nabla_u u = -\nabla p & \text{in } \Omega(t), \\ \operatorname{div}(u) = 0 & \text{in } \Omega(t), \\ p = \kappa \mathcal{A} & \text{on } \partial\Omega(t), \\ \langle u, \nu \rangle = v & \text{on } \partial\Omega(t), \\ u(0) = u_0, \end{cases} \quad (1.2)$$

where u is the fluid velocity, p is the fluid pressure, \mathcal{A} is the mean curvature of $\partial\Omega(t)$, ν is the unit outer normal to $\partial\Omega(t)$, v is the velocity of the moving boundary $\partial\Omega(t)$, and κ a non-negative constant known as coefficient of surface tension. We refer the reader to the literature (e.g. [8, 38, 61]) for a detailed discussion of these equations. It is important to point out that, despite its importance and the great deal of work dedicated to (1.2) [2, 3, 7, 6, 19, 37, 39, 44, 48, 52, 62, 64], only recently the problem has been shown to be well-posed [8, 9, 38] (other recent results, including the study of the compressible free boundary Euler equations, are [10, 11, 12, 13, 14]).

A very natural question is that of the behavior of solutions to (1.2) in the limit $\kappa \rightarrow \infty$. Physically, large values of κ correspond to domains with longer relaxation times or, more colloquially, to stiffer domains. Therefore, one would expect that solutions (1.2) with large κ should be near solutions of the standard Euler equations in the *fixed* domain $\Omega \equiv \Omega(0)$.

The study of the limit $\kappa \rightarrow \infty$, along with a proof of the corresponding convergence, was carried out by the author and David G. Ebin in [16] in the case of two-spatial dimensions (the reader is also referred to [16] for a more detailed discussion of the intuition behind this convergence). The core of the analysis consists in studying the problem from the point of view of Lagrangian coordinates, in which case all quantities can be written as time-dependent functions on the fixed domain $\Omega (= \Omega(0))$. The flow of the vector field u , $\eta(t, \cdot) : \Omega \rightarrow \mathbb{R}^n$, is decomposed in a part fixing the boundary and a boundary motion. Such decomposition takes the form

$$\eta = (\operatorname{id} + \nabla f) \circ \beta, \quad (1.3)$$

where β is a diffeomorphism of the domain Ω (so in particular $\beta(\partial\Omega) = \partial\Omega$), $f : \Omega \rightarrow \mathbb{R}$, and id is the identity diffeomorphism. The term ∇f controls the motion of the boundary, and using (1.2), it is possible to derive an equation for f . *At the linearized level and to highest order, this equation is (1.1); the third order operator $\bar{\Delta}\partial_\nu$ stems from the mean curvature of the moving boundary.* See [16] for details.

In [16], we were interested in studying the limit $\kappa \rightarrow \infty$, and therefore we relied on the aforementioned existence results for (1.2) (particularly, [8]). Therefore, the existence of solutions for the linearized problem, namely, (1.1), has not been addressed in [16]. While it will be shown in a future work that the decomposition (1.3) can be employed to derive existence of solutions to (1.2) [17], such an analysis is based on the calculus of pseudo-differential operators and techniques similar to [35]. Hence, the simpler, more traditional methods that we shall present here do not appear elsewhere.

Furthermore, the results in [17] do not cover the case of *weak* solutions to (1.1), which is the main point of this paper (see definition 1.2).

We point out that the singular limit $\kappa \rightarrow \infty$ investigated in [16, 17] fits into the larger picture of properties of solutions viewed as curves on infinite dimensional manifolds of mappings, which has been extensively studied in the context of the Euler equations. See the references [5, 18, 19, 20, 21, 22, 24, 23, 25, 26, 41], and the discussion in the introduction of [16]. While here we shall not study the dependence on the parameter κ , it is instructive to keep the above ideas in mind. In this regard, compare (1.1) with the toy-model presented in [21].

Naturally, dynamic boundary value problems have a long history, leading to variants of (1.1). Adding to the aforementioned works, whose focus is mainly on equations of hyperbolic type, the reader can consult, for instance, [27, 34] and references therein, for a point of view that stresses parabolic equations. Equations involving two time derivatives and a third order operator also have been studied before (see, for instance, [33], and references therein, and see also the related [60]). In particular, due to the elliptic operator $\overline{\Delta}$, the boundary equation (1.1b) is reminiscent of the so-called Wentzell boundary conditions, which have been widely studied by A. Favini, G. Goldstein, J. Goldstein, and S. Romanelli (a sample of such works is [28, 29, 30, 31, 32]).

In order to state our results, some notation and definitions are needed. Fix some $T > 0$. Define

$$X_T^3(\partial\Omega) = C^0([0, T], H^3(\partial\Omega)) \cap C^1([0, T], H^{\frac{3}{2}}(\partial\Omega)) \cap C^2([0, T], H^0(\partial\Omega)),$$

and

$$X_T^{\frac{3}{2}}(\partial\Omega) = C^0([0, T], H^{\frac{3}{2}}(\partial\Omega)) \cap C^1([0, T], H^0(\partial\Omega)),$$

with norms

$$\|f\|_{X_T^3(\partial\Omega)} = \sup_{t \in [0, T]} \|f\|_{3, \partial} + \sup_{t \in [0, T]} \|\dot{f}\|_{\frac{3}{2}, \partial} + \sup_{t \in [0, T]} \|\ddot{f}\|_{0, \partial},$$

and

$$\|f\|_{X_T^{\frac{3}{2}}(\partial\Omega)} = \sup_{t \in [0, T]} \|f\|_{\frac{3}{2}, \partial} + \sup_{t \in [0, T]} \|\dot{f}\|_{0, \partial}.$$

Above, $H^s(\partial\Omega)$ is the Sobolev space whose norm is denoted by $\|\cdot\|_{s, \partial}$. Notice that

$$X_T^3(\partial\Omega) \subset X_T^{\frac{3}{2}}(\partial\Omega).$$

The intersections forming $X_T^s(\partial\Omega)$ are of Sobolev spaces that differ by $\frac{3}{2}$ derivatives. This is because equation (1.1b) is second order in time and third order in space, thus each time derivative corresponds to $\frac{3}{2}$ spatial derivatives. The spaces $X_T^s(\Omega)$ are similarly defined, and the norm in $H^s(\Omega)$ is denoted $\|\cdot\|_s$.

As it is implied in the above definitions, we are working with Sobolev spaces defined with $s \in \mathbb{R}$. In the case $s \geq 0$, it is useful to have the following explicit form. Put $s = m + \sigma$, where m is an integer and $0 < \sigma < 1$. Then

$$\|u\|_s^2 = \|u\|_m^2 + [D^m u]_\sigma^2, \quad (1.4)$$

where $[\cdot]_\sigma$ is the semi-norm

$$[u]_\sigma^2 = \int_{\Omega \times \Omega} \frac{|u(x) - u(y)|^2}{|x - y|^{n+2\sigma}} dx dy,$$

with n as the dimension of Ω , and $\|\cdot\|_m^2$ as the standard Sobolev norm defined for integer m [15]. As usual, H^0 is simply the L^2 space.

As (1.1) has not appeared in the literature before, our main interest is to define a natural notion of weak solutions to problem (1.1), and then show that these solutions exist. With this in mind, our treatment will focus on the simple situation where Ω is the unit ball, and we restrict ourselves to the case $n = 3$. In this situation, problem (1.1) simplifies considerably, although many of the arguments below can be extended to a more general setting. Furthermore, this covers one of the main cases of interest, namely, that motivated by the linearization of (1.2) as discussed above and studied in [16, 17]. We now proceed to state our results.

Denote $L^2(T) = L^2([0, T] \times \partial\Omega)$. Define a map

$$\begin{aligned}\mathcal{L} : X_T^3(\partial\Omega) &\rightarrow L^2(T) \times H^{\frac{3}{2}}(\partial\Omega) \times H^0(\partial\Omega) \equiv \mathcal{H} \\ \mathcal{L}(f) &= (\ddot{f} - \kappa \bar{\Delta} \partial_\nu f, f(0), \dot{f}(0)),\end{aligned}$$

where $\partial_\nu f$ is computed using the harmonic extension of f to Ω , and we write $f(0) = f(0, \cdot)$, $\dot{f}(0) = \dot{f}(0, \cdot)$. Let $\mathcal{R} \subset \mathcal{H}$ be the image of \mathcal{L} . We shall prove the following.

Proposition 1.1. *Let $\Omega \subset \mathbb{R}^3$ be the ball of radius one centered at the origin, and fix some $T > 0$. Let \mathcal{L} , \mathcal{R} , and \mathcal{H} be as above. Then:*

(i) \mathcal{L} is injective, and since $X_T^3(\partial\Omega) \subset X_T^{\frac{3}{2}}(\partial\Omega)$, \mathcal{L}^{-1} defines a map

$$\mathcal{L}^{-1} : \mathcal{R} \rightarrow X_T^{\frac{3}{2}}(\partial\Omega).$$

The map \mathcal{L}^{-1} is continuous as a map from \mathcal{R} to $X_T^{\frac{3}{2}}(\partial\Omega)$.

(ii) The closure of \mathcal{R} in \mathcal{H} , denoted $\overline{\mathcal{R}}$, is the whole of \mathcal{H} , i.e., $\overline{\mathcal{R}} = \mathcal{H}$, and \mathcal{L}^{-1} extends to a continuous linear map, $\overline{\mathcal{L}^{-1}}$, from \mathcal{H} to $X_T^{\frac{3}{2}}(\partial\Omega)$.

(iii) The image of $\overline{\mathcal{L}^{-1}}$ is

$$\overline{X_T^3(\partial\Omega)}^{X_T^{\frac{3}{2}}(\partial\Omega)},$$

i.e., the closure of $X_T^3(\partial\Omega) \subset X_T^{\frac{3}{2}}(\partial\Omega)$ in the $X_T^{\frac{3}{2}}(\partial\Omega)$ topology.

We can now introduce the following.

Definition 1.2. Let $f_0 \in H^{\frac{3}{2}}(\partial\Omega)$, $f_1 \in H^0(\partial\Omega)$, and $G \in L^2(T)$ be given. We say that

$$f \in \overline{X_T^3(\partial\Omega)}^{X_T^{\frac{3}{2}}(\partial\Omega)} \cap X_T^2(\Omega)$$

is a weak solution of (1.1), if $f|_{\partial\Omega} = \overline{\mathcal{L}^{-1}}((G, f_0, f_1))$, where $\overline{\mathcal{L}^{-1}} : \mathcal{H} \rightarrow \overline{X_T^3(\partial\Omega)}^{X_T^{\frac{3}{2}}(\partial\Omega)}$ is the map given by proposition 1.1, and f satisfies (1.1a) in Ω .

To understand why this is a suitable definition of weak solutions for problem (1.1), one should think of the example of the wave equation. In that case, given initial data in $H^1(\mathbb{R}^n) \times H^0(\mathbb{R}^n)$ and an inhomogeneous term in, say, $L^1([0, T], H^0(\mathbb{R}^n))$, the weak solution u is in $C^0([0, T], H^1(\mathbb{R}^n)) \cap C^1([0, T], H^0(\mathbb{R}^n))$. Thus, the weak solution has one less spatial derivative than the order of the equation, with $\partial_t u$ one degree less differentiable in space than u itself. Such a regularity is a consequence of the energy estimate, in which an integration by parts is performed. In our case, each time derivative corresponds to $\frac{3}{2}$ spatial ones, and we heuristically think of integrating by parts half of the derivatives of the third order spatial term.

Proposition 1.1 essentially contains the existence of weak solutions, but we state it separately for convenience.

Theorem 1.3. *Let $\Omega \subset \mathbb{R}^3$ be the ball of radius one centered at the origin, and fix some $T > 0$. Given $G \in L^2(T)$, $f_0 \in H^{\frac{3}{2}}(\partial\Omega)$, and $f_1 \in H^0(\partial\Omega)$, there exists a unique weak solution to the problem (1.1).*

It should be stressed that for sufficiently regular data, existence for (1.1) can probably be derived by other means. The novelty of theorem 1.3 is centered around the notion of weak solutions and their existence. In this regard, it is important to stress that a semi-group approach can also be employed to study (1.1), in which case one is led, via Stone's theorem, to investigate the existence of mild-solutions to the problem [4, 63]. Such mild solutions are, in fact, closely related to our notion of weak solution. We believe, however, that the energy-method approach here employed is significantly simpler in the sense that it does not rely on heavy functional-analytic techniques, and is also of independent interest to the community more acquainted with such type of estimates.

2. ENERGY ESTIMATES.

In this section we carry out the necessary energy estimates for the proofs of proposition 1.1 and theorem 1.3. We start recalling some useful tools and fixing some of the notation.

2.1. Auxiliary results. Here, we collect some well known facts that will be used in the paper. Their proofs can be found in many sources, e.g., [1, 5, 15, 20, 46].

First, recall that restriction to the boundary gives rise to a bounded linear map,

$$\|u\|_{s,\partial} \leq C \|u\|_{s+\frac{1}{2}}, \quad s > 0, \quad (2.1)$$

with $C = C(n, s, \Omega)$.

The usual interpolation inequality will also be needed: if $s_1 < s_2 < s_3$, then

$$\|u\|_{s_2} \leq \|u\|_{s_1}^{\frac{s_3-s_2}{s_3-s_1}} \|u\|_{s_3}^{\frac{s_2-s_1}{s_3-s_1}}. \quad (2.2)$$

We finally recall the standard Cauchy inequality with γ ,

$$ab \leq \gamma a^2 + \frac{1}{4\gamma} b^2, \quad (2.3)$$

$\gamma > 0$ (this inequality is usually called Cauchy inequality with ε , with the letter ε used instead of γ . We shall reserve ε for other purposes below, thus we use γ in (2.3) to avoid confusion).

2.2. Coordinates and notation. Here, we make some remarks about coordinates and notation.

Recalling that Ω is the ball of radius one centered at the origin, we write

$$\partial\Omega_r = \partial B_r(0).$$

Sometimes, we employ spherical coordinates (r, ϕ, θ) , so that

$$\Delta = \partial_r^2 + \frac{2}{r} \partial_r + \frac{1}{r^2} \Delta_{S^2}, \quad (2.4)$$

where Δ_{S^2} is the Laplacian on the standard round sphere, given in these coordinates by

$$\Delta_{S^2} = \partial_\phi^2 + \frac{\cos \phi}{\sin \phi} \partial_\phi + \frac{1}{\sin^2 \phi} \partial_\theta^2.$$

In particular, the Laplacian on $\partial\Omega_r$, which we denote $\overline{\Delta}$ for any r , is

$$\overline{\Delta} = \frac{1}{r^2} \Delta_{S^2}. \quad (2.5)$$

We shall use $\overline{\Delta}$ as an operator on the whole of Ω . To be precise, this is not defined at zero, but the origin can be removed without changing the value of the integrals \int_{Ω} containing $\overline{\Delta}$ that will appear below. In particular, $\overline{\Delta}f$ is defined on Ω (but the origin).

We shall also make use of the following coordinate choice. For $\varepsilon > 0$, let

$$\Omega_{\varepsilon} = \Omega \setminus (B_{\varepsilon}(0) \cup C_{\varepsilon}),$$

where C_{ε} is the cone given in spherical coordinates by $\{\phi \geq \pi - \varepsilon\}$. Choose Fermi coordinates $\{x^{\mu}\}_{\mu=1}^3$ at the north pole of $\partial\Omega$. These coordinates cover Ω_{ε} , and the Euclidean metric takes the form $g = (g_{\alpha\beta})$, with $g_{33} = 1$, $g_{i3} = 0$, $i = 1, 2$, and g_{ij} , $i, j = 1, 2$, being the metric induced on the level sets $\{x^3 = \text{constant}\}$, which in turn correspond to $\partial\Omega_r \cap \Omega_{\varepsilon}$. Furthermore, ∂_3 is orthogonal to $\partial\Omega_r \cap \Omega_{\varepsilon}$, and $\partial_3 = -\partial_r$. We illustrate the construction of these coordinates in figure 1, where we also depict further notation that will be used below.

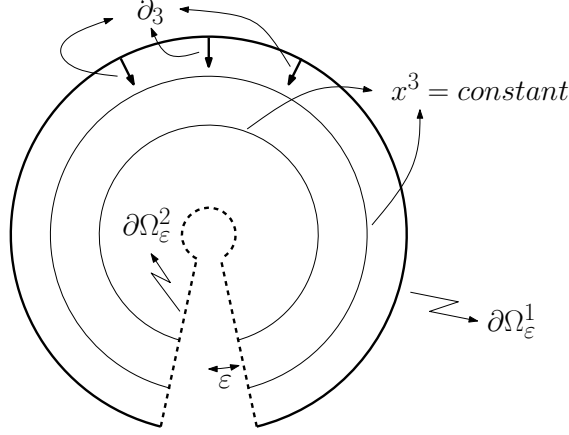


FIGURE 1. Illustration of the set Ω_{ε} and its boundary $\partial\Omega_{\varepsilon} = \partial\Omega_{\varepsilon}^1 \cup \partial\Omega_{\varepsilon}^2$.

For the rest of the paper, the following convention is adopted.

Notation 2.1. Greek indices run from 1 to 3 and Latin indices from 1 to 2. The letter C will be used to denote several different constants, as usual.

In the above coordinates, equation (1.1a) then reads

$$\nabla^{\mu} \nabla_{\mu} f = 0,$$

where ∇ is covariant differentiation in the Euclidean metric, but written in this system of coordinates. Notice that all covariant derivatives will commute, as the metric is flat, and we shall use this in the calculations below.

2.3. Basic energy inequality. For the rest of this section, let $f \in X_T^3(\partial\Omega)$ be a solution to (1.1b)-(1.1c). We also denote by f its harmonic extension to Ω , i.e., f satisfies (1.1a) in Ω .

Define the energy

$$E = \frac{1}{2} \int_{\partial\Omega} \left(\dot{f}^2 - \kappa f \overline{\Delta} \partial_{\nu} f \right). \quad (2.6)$$

Differentiating and integrating by parts,

$$\dot{E} = \int_{\partial\Omega} \dot{f} \ddot{f} - \frac{1}{2} \kappa \int_{\partial\Omega} \dot{f} \overline{\Delta} \partial_{\nu} f - \frac{1}{2} \kappa \int_{\partial\Omega} \overline{\Delta} f \partial_{\nu} \dot{f}. \quad (2.7)$$

Let $\varphi = \varphi(r)$ be a sufficient regular function such that $\varphi(1) = 1$, i.e., $\varphi \equiv 1$ on $\partial\Omega$. Apply Green's identity

$$\int_{\Omega} (u\Delta v - v\Delta u) = \int_{\partial\Omega} \left(u \frac{\partial v}{\partial \nu} - v \frac{\partial u}{\partial \nu} \right)$$

with $v = \varphi \bar{\Delta} f$ (which equals $\bar{\Delta} f$ on $\partial\Omega$) and $u = \dot{f}$, to get

$$\int_{\partial\Omega} \bar{\Delta} f \partial_{\nu} \dot{f} = \int_{\partial\Omega} \varphi \bar{\Delta} f \partial_{\nu} \dot{f} = \int_{\partial\Omega} \dot{f} \bar{\Delta} \partial_{\nu} f + \int_{\partial\Omega} \dot{f} [\partial_{\nu}, \varphi \bar{\Delta}] f - \int_{\Omega} \dot{f} \Delta(\varphi \bar{\Delta} f), \quad (2.8)$$

where $[\cdot, \cdot]$ is the commutator, we have used that (1.1a) implies $\Delta \dot{f} = 0$, and that $\varphi = 1$ on $\partial\Omega$. Using (2.8) into (2.7) gives

$$\dot{E} = \int_{\partial\Omega} \dot{f} \left(\ddot{f} - \kappa \bar{\Delta} \partial_{\nu} f \right) + r_1 \quad (2.9)$$

where

$$r_1 = -\frac{1}{2} \kappa \int_{\partial\Omega} \dot{f} [\partial_{\nu}, \varphi \bar{\Delta}] f + \frac{1}{2} \kappa \int_{\Omega} \dot{f} \Delta(\varphi \bar{\Delta} f).$$

To analyze r_1 , pick $\varphi(r) = r^2$, which satisfies the previous assumptions on φ . Then, on $\partial\Omega$,

$$\partial_{\nu}(\varphi \bar{\Delta} f) = \partial_r(r^2 \bar{\Delta} f) = \partial_r \Delta_{S^2} f = \Delta_{S^2} \partial_r f = \bar{\Delta} \partial_{\nu} f = \varphi \bar{\Delta} \partial_{\nu} f,$$

which implies $[\partial_{\nu}, \varphi \bar{\Delta}] = 0$. Using also (2.4),

$$\Delta(\varphi \bar{\Delta} f) = \Delta \Delta_{S^2} f = \Delta_{S^2} \partial_r^2 f + \Delta_{S^2} \left(\frac{2}{r} \partial_r f \right) + \Delta_{S^2} \left(\frac{1}{r^2} \Delta_{S^2} f \right) = \Delta_{S^2} \Delta f = 0.$$

Thus $r_1 = 0$ and (2.9) becomes

$$\dot{E} = \int_{\partial\Omega} \dot{f} \left(\ddot{f} - \kappa \bar{\Delta} \partial_{\nu} f \right). \quad (2.10)$$

Invoking (1.1b) and the Cauchy-Schwarz inequality, (2.10) gives

$$\dot{E} \leq \frac{1}{2} \|\dot{f}\|_{0,\partial}^2 + \frac{1}{2} \|G\|_{0,\partial}^2, \quad (2.11)$$

where we recall that $\|\cdot\|_{s,\partial}$ is the Sobolev norm on the boundary.

Lemma 2.2.

$$-\int_{\partial\Omega} \bar{\Delta} f \partial_{\nu} f \geq 0.$$

Proof. Integrating by parts,

$$-\int_{\partial\Omega} \bar{\Delta} f \partial_{\nu} f = \int_{\partial\Omega} \langle \nabla_{\partial} f, \nabla_{\partial} \partial_{\nu} f \rangle, \quad (2.12)$$

where ∇_{∂} is the gradient on $\partial\Omega$. We need to commute ∇_{∂} and ∂_{ν} . This is easily done in spherical coordinates, yielding

$$[\nabla_{\partial}, \nabla_{\nu}] = \frac{1}{r} \nabla_{\partial}.$$

(2.12) becomes

$$-\int_{\partial\Omega} \bar{\Delta} f \partial_{\nu} f = \int_{\partial\Omega} \frac{1}{r} |\nabla_{\partial} f|^2 + \int_{\partial\Omega} \langle \nabla_{\partial} f, \nabla_{\nu} \nabla_{\partial} f \rangle. \quad (2.13)$$

Choose Fermi coordinates as explained at the beginning of this section. Then $\nabla^\mu \nabla_\mu f = 0$ implies $\nabla^\mu \nabla_\mu \nabla_\sigma f = 0$, and $\nabla^\sigma f \nabla^\mu \nabla_\mu \nabla_\sigma f = 0$. Integrating over Ω_ε and integrating by parts,

$$0 = \int_{\Omega_\varepsilon} \nabla^\sigma f \nabla^\mu \nabla_\mu \nabla_\sigma f = - \int_{\Omega_\varepsilon} \nabla^\mu \nabla^\sigma f \nabla_\mu \nabla_\sigma f + \int_{\partial\Omega_\varepsilon} \nabla^\sigma f \nabla_V \nabla_\sigma f,$$

so

$$\int_{\Omega_\varepsilon} \nabla^\mu \nabla^\sigma f \nabla_\mu \nabla_\sigma f = \int_{\partial\Omega_\varepsilon^1} \nabla^\sigma f \nabla_V \nabla_\sigma f + \int_{\partial\Omega_\varepsilon^2} \nabla^\sigma f \nabla_V \nabla_\sigma f, \quad (2.14)$$

where $\partial\Omega_\varepsilon^1 = \partial\Omega \setminus (\partial\Omega_\varepsilon \cap \partial\Omega)$, $\partial\Omega_\varepsilon^2 = \partial\Omega_\varepsilon \setminus \partial\Omega_\varepsilon^1$ (see figure 1), and ∇_V is covariant differentiation in the normal direction, with V the unit outer normal. On $\partial\Omega_\varepsilon^1$, $\nabla_V = -\nabla_3$. Compute

$$\nabla_3 \nabla_\sigma f = \partial_3 \partial_\sigma f - \Gamma_{3\sigma}^j \partial_j f,$$

where we used that any Christoffel symbol with two indices 3 vanishes in Fermi coordinates. Let $\bar{\nabla}$ be the covariant derivative on $\partial\Omega$. Then, since $\nabla_i f = \partial_i f = \bar{\nabla}_i f$,

$$\nabla_3 \nabla_i f = \nabla_3 \bar{\nabla}_i f.$$

Hence,

$$\int_{\partial\Omega_\varepsilon^1} \nabla^\sigma f \nabla_V \nabla_\sigma f = - \int_{\partial\Omega_\varepsilon^1} \bar{\nabla}^i f \nabla_3 \bar{\nabla}_i f - \int_{\partial\Omega_\varepsilon^1} \nabla^3 f \nabla_3 \nabla_3 f \quad (2.15)$$

Using (2.15) into (2.14) gives

$$\int_{\Omega_\varepsilon} \nabla^\mu \nabla^\sigma f \nabla_\mu \nabla_\sigma f = - \int_{\partial\Omega_\varepsilon^1} \bar{\nabla}^i f \nabla_3 \bar{\nabla}_i f - \int_{\partial\Omega_\varepsilon^1} \nabla^3 f \nabla_3 \nabla_3 f + \int_{\partial\Omega_\varepsilon^2} \nabla^\sigma f \nabla_V \nabla_\sigma f, \quad (2.16)$$

Taking ∇_3 of $\nabla^\mu \nabla_\mu f = 0$, commuting the covariant derivatives, multiplying by $\nabla^3 f$ and integrating by parts,

$$0 = \int_{\Omega_\varepsilon} \nabla^3 f \nabla^\mu \nabla_\mu \nabla_3 f = - \int_{\Omega_\varepsilon} \nabla^\mu \nabla^3 f \nabla_\mu \nabla_3 f + \int_{\partial\Omega_\varepsilon} \nabla^3 f \nabla_V \nabla_3 f,$$

so, since $\nabla_V = -\nabla_3$ on $\partial\Omega_\varepsilon^1$,

$$\int_{\partial\Omega_\varepsilon^1} \nabla^3 f \nabla_3 \nabla_3 f = - \int_{\Omega_\varepsilon} \nabla^\mu \nabla^3 f \nabla_\mu \nabla_3 f + \int_{\partial\Omega_\varepsilon^2} \nabla^3 f \nabla_V \nabla_3 f$$

Using the above expression for $\int_{\partial\Omega_\varepsilon^1} \nabla^3 f \nabla_3 \nabla_3 f$ into (2.16),

$$\begin{aligned} \int_{\Omega_\varepsilon} \nabla^\mu \nabla^\sigma f \nabla_\mu \nabla_\sigma f &= - \int_{\partial\Omega_\varepsilon^1} \bar{\nabla}^i f \nabla_3 \bar{\nabla}_i f + \int_{\Omega_\varepsilon} \nabla^\mu \nabla^3 f \nabla_\mu \nabla_3 f \\ &\quad - \int_{\partial\Omega_\varepsilon^2} \nabla^3 f \nabla_V \nabla_3 f + \int_{\partial\Omega_\varepsilon^2} \nabla^\sigma f \nabla_V \nabla_\sigma f, \end{aligned}$$

so that

$$\begin{aligned} \int_{\partial\Omega_\varepsilon^1} \bar{\nabla}^i f \nabla_V \bar{\nabla}_i f &= - \int_{\partial\Omega_\varepsilon^1} \bar{\nabla}^i f \nabla_3 \bar{\nabla}_i f = \int_{\Omega_\varepsilon} \nabla^\mu \nabla^\sigma f \nabla_\mu \nabla_\sigma f - \int_{\Omega_\varepsilon} \nabla^\mu \nabla^3 f \nabla_\mu \nabla_3 f \\ &\quad + \int_{\partial\Omega_\varepsilon^2} \nabla^3 f \nabla_V \nabla_3 f - \int_{\partial\Omega_\varepsilon^2} \nabla^\sigma f \nabla_V \nabla_\sigma f. \end{aligned} \quad (2.17)$$

The first two terms on the right hand side combine to give

$$\int_{\Omega_\varepsilon} (\nabla^\mu \nabla^\sigma f \nabla_\mu \nabla_\sigma f - \nabla^\mu \nabla^3 f \nabla_\mu \nabla_3 f) = \int_{\Omega_\varepsilon} (|\nabla \nabla f|^2 - |\nabla \nabla_V f|^2) \geq 0.$$

For the integrals along $\partial\Omega_\varepsilon^2$, write $\partial\Omega_\varepsilon^2 = \partial\Omega_\varepsilon^{21} \cup \partial\Omega_\varepsilon^{22}$, where

$$\partial\Omega_\varepsilon^{21} = \partial\Omega_\varepsilon^2 \cap \partial B_\varepsilon(0),$$

and

$$\partial\Omega_\varepsilon^{22} = \partial\Omega_\varepsilon^2 \setminus \partial\Omega_\varepsilon^{21}.$$

Then

$$\begin{aligned} \left| \int_{\partial\Omega_\varepsilon^{21}} \nabla^3 f \nabla_V \nabla_3 f - \int_{\partial\Omega_\varepsilon^{21}} \nabla^\sigma f \nabla_V \nabla_\sigma f \right| &\leq \int_{\partial\Omega_\varepsilon^{21}} |\nabla^3 f \nabla_V \nabla_3 f| + \int_{\partial\Omega_\varepsilon^{21}} |\nabla^\sigma f \nabla_V \nabla_\sigma f| \\ &\leq \int_{\partial B_\varepsilon(0)} |\nabla^3 f \nabla_V \nabla_3 f| + \int_{\partial B_\varepsilon(0)} |\nabla^\sigma f \nabla_V \nabla_\sigma f| \\ &\leq C \int_{\partial B_\varepsilon(0)} \|f\|_{2, \partial B_\varepsilon(0)}^2 \\ &\leq C \int_{\partial B_\varepsilon(0)} \|f\|_{\frac{5}{2}}^2 \\ &\leq C \|f\|_{\frac{5}{2}}^2 \varepsilon^2, \end{aligned} \tag{2.18}$$

where $\|\cdot\|_{2, \partial B_\varepsilon(0)}$ is the Sobolev norm on $\partial B_\varepsilon(0)$, and on the next-to-the-last step we used (2.1). Similarly,

$$\left| \int_{\partial\Omega_\varepsilon^{22}} \nabla^3 f \nabla_V \nabla_3 f - \int_{\partial\Omega_\varepsilon^{22}} \nabla^\sigma f \nabla_V \nabla_\sigma f \right| \leq C \|f\|_{\frac{5}{2}}^2 \varepsilon. \tag{2.19}$$

From (2.18) and (2.19) we see that the integrals over $\partial\Omega_\varepsilon^2$ vanish in the limit $\varepsilon \rightarrow 0^+$. Hence,

$$\int_{\partial\Omega} \langle \nabla_\partial f, \nabla_\nu \nabla_\partial f \rangle = \lim_{\varepsilon \rightarrow 0^+} \int_{\partial\Omega_\varepsilon^1} \bar{\nabla}^i f \nabla_V \bar{\nabla}_i f \geq 0, \tag{2.20}$$

where we recall that ν is the outer unit normal to $\partial\Omega$. Combining (2.20) with (2.13) yields the result. \square

As a consequence of lemma 2.2, we have $E(t) \geq 0$, and

$$\|\dot{f}\|_{0, \partial}^2 \leq E. \tag{2.21}$$

Thus, (2.11) gives

$$\dot{E} \leq \frac{1}{2} \|G\|_{0, \partial}^2 + \frac{1}{2} E,$$

or,

$$E(t) \leq E(0) + \frac{1}{2} \int_0^t \|G(\tau)\|_{0, \partial}^2 d\tau + \frac{1}{2} \int_0^t E(\tau) d\tau,$$

which gives, after iteration,

$$E(t) \leq \left(E(0) + \frac{1}{2} \int_0^t \|G(\tau)\|_{0, \partial}^2 d\tau \right) e^{\frac{1}{2}t}. \tag{2.22}$$

From (1.1a) and standard elliptic theory we have

$$\|f\|_s \leq C \|f\|_{s-\frac{1}{2}, \partial}. \tag{2.23}$$

And invoking elliptic theory once more, (2.23) then gives the following estimate:

$$\begin{aligned}
\left| \int_{\partial\Omega} f \overline{\Delta} \partial_\nu f \right| &= \left| \int_{\partial\Omega} \overline{\Delta} f \partial_\nu f \right| \\
&\leq \| \overline{\Delta} f \|_{-\frac{1}{2}, \partial} \| \partial_\nu f \|_{\frac{1}{2}, \partial} \\
&\leq C \| f \|_{\frac{3}{2}, \partial} \| f \|_2 \\
&\leq C \| f \|_{\frac{3}{2}, \partial}^2.
\end{aligned} \tag{2.24}$$

Thus, (2.24) and the definition of E , i.e. (2.6), imply

$$E \leq C \| f \|_{\frac{3}{2}, \partial}^2 + C \| \dot{f} \|_{0, \partial}^2. \tag{2.25}$$

Combining (2.25) at time zero with (2.22) produces

$$E(t) \leq C \left(\| f(0) \|_{\frac{3}{2}, \partial}^2 + C \| \dot{f}(0) \|_{0, \partial}^2 + \frac{1}{2} \int_0^t \| G(\tau) \|_{0, \partial}^2 d\tau \right) e^{\frac{1}{2}t}. \tag{2.26}$$

Next, we show that

Lemma 2.3.

$$- \int_{\partial\Omega} f \overline{\Delta} \partial_\nu f = - \int_{\partial\Omega} \overline{\Delta} f \partial_\nu f \geq C \| f \|_{\frac{3}{2}, \partial}^2 - C \| f \|_{0, \partial}^2.$$

Remark 2.4. The presence of the negative term $- \| f \|_{0, \partial}^2$ is necessary as $\int_{\partial\Omega} \overline{\Delta} f \partial_\nu f$ is zero on constant functions.

Proof. The equality follows by integration by parts. In light of (2.13), it suffices to obtain the inequality for

$$\int_{\partial\Omega} \langle \nabla_\partial f, \nabla_\nu \nabla_\partial f \rangle,$$

which, as in lemma 2.2, along $\partial\Omega_\varepsilon^1$ corresponds to the term

$$- \int_{\partial\Omega_\varepsilon^1} \overline{\nabla}^i f \nabla_3 \overline{\nabla}_i f.$$

Proceed as in lemma 2.2 until (2.17), and consider its first two terms on the right hand side. Their integrand gives

$$\nabla^\mu \nabla^\sigma f \nabla_\mu \nabla_\sigma f - \nabla^\mu \nabla^3 f \nabla_\mu \nabla_3 f = \nabla^\mu \nabla^i f \nabla_\mu \nabla_i f.$$

Letting $\overline{\nabla}$ denote the covariant derivative along $\partial\Omega_r \cap \Omega_\varepsilon$ (which correspond to the level sets $x^3 = \text{constant}$), noticing that $\overline{\nabla}_i f = \partial_i f = \nabla_i f$ and that, therefore, $\overline{\nabla}_i$ gives a well-defined operator on Ω_ε , we can write the above as

$$\nabla^\mu \nabla^\sigma f \nabla_\mu \nabla_\sigma f - \nabla^\mu \nabla^3 f \nabla_\mu \nabla_3 f = |\nabla \overline{\nabla} f|^2,$$

thus (2.17) gives,

$$- \int_{\partial\Omega_\varepsilon^1} \overline{\nabla}^i f \nabla_3 \overline{\nabla}_i f = \int_{\Omega_\varepsilon} |\nabla \overline{\nabla} f|^2 + \int_{\partial\Omega_\varepsilon^2} \nabla^3 f \nabla_V \nabla_3 f - \int_{\partial\Omega_\varepsilon^2} \nabla^\sigma f \nabla_V \nabla_\sigma f. \tag{2.27}$$

But

$$\int_{\Omega_\varepsilon} |\nabla \overline{\nabla} f|^2 \geq \| \overline{\nabla} f \|_1^2 - C \| f \|_1^2 \geq \| \overline{\nabla} f \|_1^2 - C \| f \|_{\frac{1}{2}, \partial_\varepsilon}^2 \geq C \| \overline{\nabla} f \|_{\frac{1}{2}, \partial_\varepsilon}^2 - C \| f \|_{\frac{1}{2}, \partial_\varepsilon}^2, \tag{2.28}$$

where in the last step we used (2.1), in the next-to-the-last, (2.23), and $\|\cdot\|_{s,\partial_\varepsilon}$ is the Sobolev norm on the boundary $\partial\Omega_\varepsilon$. Applying the interpolation inequality (2.2) with $s_1 = 0$, $s_2 = \frac{1}{2}$, and $s_3 = 1$,

$$\|f\|_{\frac{1}{2},\partial_\varepsilon}^2 \leq \frac{C}{\gamma} \|f\|_{0,\partial_\varepsilon}^2 + \gamma \|f\|_{1,\partial_\varepsilon}^2, \quad (2.29)$$

where the Cauchy inequality with γ , (2.3), has been employed. Using (2.29) in (2.28) and recalling (1.4),

$$\begin{aligned} \int_{\Omega_\varepsilon} |\nabla \bar{\nabla} f|^2 &\geq C \|\bar{\nabla} f\|_{\frac{1}{2},\partial_\varepsilon}^2 - C\gamma \|f\|_{1,\partial_\varepsilon}^2 - \frac{C}{\gamma} \|f\|_{0,\partial_\varepsilon}^2 \\ &= C [\bar{\nabla} f]_{\frac{1}{2},\partial_\varepsilon}^2 + C \|f\|_{1,\partial_\varepsilon}^2 - C\gamma \|f\|_{1,\partial_\varepsilon}^2 - \frac{C}{\gamma} \|f\|_{0,\partial_\varepsilon}^2 \\ &\geq C [\bar{\nabla} f]_{\frac{1}{2},\partial_\varepsilon}^2 + C \|f\|_{1,\partial_\varepsilon}^2 - \frac{C}{\gamma} \|f\|_{0,\partial_\varepsilon}^2, \end{aligned} \quad (2.30)$$

where the last step follows by choosing γ sufficiently small. Since $\bar{\nabla}$ is differentiation along $\partial\Omega_\varepsilon$, recalling (1.4) once more, we see that

$$C [\bar{\nabla} f]_{\frac{1}{2},\partial_\varepsilon}^2 + C \|f\|_{1,\partial_\varepsilon}^2 \geq C \|f\|_{\frac{3}{2},\partial_\varepsilon}^2,$$

so that (2.30) gives

$$\int_{\Omega_\varepsilon} |\nabla \bar{\nabla} f|^2 \geq C \|f\|_{\frac{3}{2},\partial_\varepsilon}^2 - C \|f\|_{0,\partial_\varepsilon}^2,$$

and therefore (2.27) implies

$$-\int_{\partial\Omega_\varepsilon^1} \bar{\nabla}^i f \nabla_3 \bar{\nabla}_i f \geq C \|f\|_{\frac{3}{2},\partial_\varepsilon}^2 - C \|f\|_{0,\partial_\varepsilon}^2 + \int_{\partial\Omega_\varepsilon^2} \nabla^3 f \nabla_V \nabla_3 f - \int_{\partial\Omega_\varepsilon^2} \nabla^\sigma f \nabla_V \nabla_\sigma f. \quad (2.31)$$

To finish the proof, split the first two terms on the right hand of (2.31) in integrals along $\partial\Omega_\varepsilon^{21}$ and $\partial\Omega_\varepsilon^{22}$. Arguing as in lemma 2.2, all integrals on $\partial\Omega_\varepsilon^2$ vanish in the limit $\varepsilon \rightarrow 0^+$, which gives the result. \square

As a consequence of lemma 2.3 and the definition 2.6, we have

$$\|f\|_{\frac{3}{2},\partial}^2 \leq \frac{C}{\kappa} E + C \|f\|_{0,\partial}^2. \quad (2.32)$$

Using the fundamental theorem of calculus and the Cauchy-Schwarz inequality,

$$\|f\|_{0,\partial}^2 \leq C \|f(0)\|_{0,\partial}^2 + C \left(\int_0^t \|\dot{f}\|_{0,\partial} \right)^2 \leq C \|f(0)\|_{0,\partial}^2 + Ct \int_0^t \|\dot{f}\|_{0,\partial}^2. \quad (2.33)$$

In the above, we used Jensen's inequality,

$$h\left(\int f\right) \leq \int h(f),$$

where h is a convex function and \int the average over the domain of integration, to estimate

$$\left(\int_0^t \|\dot{f}\|_{0,\partial} \right)^2 = t^2 \left(\int_0^t \|\dot{f}\|_{0,\partial} \right)^2 \leq t \int_0^t \|\dot{f}\|_{0,\partial}^2.$$

Thus, (2.21), (2.32), and (2.33) give

$$\|f\|_{\frac{3}{2},\partial}^2 \leq \frac{C}{\kappa} E + C \|f(0)\|_{0,\partial}^2 + Ct \int_0^t E. \quad (2.34)$$

Inequalities (2.21) and (2.34) combined with (2.26) give bounds for $\|\dot{f}\|_{0,\partial}$ and $\|f\|_{\frac{3}{2},\partial}^2$. More precisely, we have

$$\|\dot{f}\|_{0,\partial}^2 \leq C \left(\|f(0)\|_{\frac{3}{2},\partial}^2 + \|\dot{f}(0)\|_{0,\partial}^2 + \frac{1}{2} \int_0^t \|G(\tau)\|_{0,\partial}^2 d\tau \right) e^{\frac{1}{2}t}, \quad (2.35)$$

and

$$\begin{aligned} \|f\|_{\frac{3}{2},\partial}^2 &\leq C \|f(0)\|_{0,\partial}^2 + \frac{C}{\kappa} \left(\|f(0)\|_{\frac{3}{2},\partial}^2 + \|\dot{f}(0)\|_{0,\partial}^2 + \frac{1}{2} \int_0^t \|G(\tau)\|_{0,\partial}^2 d\tau \right) e^{\frac{1}{2}t} \\ &\quad + Ct^2 e^{\frac{1}{2}T} \left(\|f(0)\|_{\frac{3}{2},\partial}^2 + \|\dot{f}(0)\|_{0,\partial}^2 + \int_0^t \|G(\tau)\|_{0,\partial}^2 d\tau \right), \end{aligned} \quad (2.36)$$

where we used $e^{\frac{1}{2}t} \leq e^{\frac{1}{2}T}$, and

$$\int_0^t \int_0^\tau \|G(\sigma)\|_{0,\partial}^2 d\sigma d\tau \leq \int_0^t \int_0^t \|G(\sigma)\|_{0,\partial}^2 d\sigma d\tau = t \int_0^t \|G(\sigma)\|_{0,\partial}^2 d\sigma.$$

3. PROOFS.

We are now ready to proof proposition 1.1 and theorem 1.3. Let $X_T^{3,+}(\partial\Omega)$ be the subspace of $X_T^3(\partial\Omega)$ consisting of functions such that $f(0) = \dot{f}(0) = 0$, and $X_T^{3,-}(\partial\Omega)$ be the subspace of $X_T^3(\partial\Omega)$ consisting of functions such that $f(T) = \dot{f}(T) = 0$. Recall that $X_T^3(\partial\Omega) \subset X_T^{\frac{3}{2}}(\partial\Omega)$.

Lemma 3.1. $X_T^{3,\pm}(\partial\Omega)$ is dense in $L^2(T)$.

Proof. This is very similar to the proof that compactly supported functions are dense in Sobolev spaces, using mollifiers. \square

Proof of proposition 1.1-(i): From (2.35) and (2.36) with $G = 0$, $f(0) = 0$, and $\dot{f}(0) = 0$, it follows that \mathcal{L} is injective.

Let $(G, f_0, f_1) \in \mathcal{H}$ and $f = \mathcal{L}^{-1}((G, f_0, f_1))$. Then (2.36) gives

$$\|f\|_{\frac{3}{2},\partial} \leq C(T, \kappa) \left(\|f_0\|_{\frac{3}{2},\partial} + \|f_1\|_{0,\partial} + \sqrt{\int_0^T \|G(\tau)\|_{0,\partial}^2 d\tau} \right),$$

where $C(T, \kappa)$ is a constant that depends on T and κ , and we have used that

$$\int_0^t \|G(\tau)\|_{0,\partial}^2 d\tau \leq \int_0^T \|G(\tau)\|_{0,\partial}^2 d\tau, \quad t \in [0, T],$$

and that the exponential is an increasing function. But

$$\int_0^T \|G(\tau)\|_{0,\partial}^2 d\tau = \int_0^T \int_\Omega |G(\tau, x)|^2 dx d\tau \leq C \|G\|_{L^2(T)}^2,$$

so that

$$\|f\|_{\frac{3}{2},\partial} \leq C(T, \kappa) \left(\|f_0\|_{\frac{3}{2},\partial} + \|f_1\|_{0,\partial} + \|G\|_{L^2(T)} \right).$$

Similarly, from (2.35) we also conclude,

$$\|\dot{f}\|_{0,\partial} \leq C(T, \kappa) \left(\|f_0\|_{\frac{3}{2},\partial} + \|f_1\|_{0,\partial} + \|G\|_{L^2(T)} \right).$$

These last two inequalities imply

$$\|f\|_{X_T^{\frac{3}{2}}(\partial\Omega)} \leq C(T, \kappa) \left(\|f_0\|_{\frac{3}{2}, \partial} + \|f_1\|_{0, \partial} + \|G\|_{L^2(T)} \right). \quad (3.1)$$

As the right hand side of this last inequality is the norm of (G, f_0, f_1) in the topology of \mathcal{H} , we conclude that \mathcal{L}^{-1} is a continuous map. \square

Proof of proposition 1.1-(ii): Denoting by $\overline{\mathcal{R}}$ the closure of \mathcal{R} in \mathcal{H} , \mathcal{L}^{-1} extends, by continuity, to a continuous linear map

$$\overline{\mathcal{L}^{-1}} : \overline{\mathcal{R}} \rightarrow X_T^{\frac{3}{2}}(\partial\Omega).$$

In order to show that the closure of \mathcal{R} in \mathcal{H} is the whole of \mathcal{H} , i.e.,

$$\overline{\mathcal{R}} = \mathcal{H},$$

we will prove that if $v \perp \overline{\mathcal{R}}$, then $v = 0$. As \mathcal{R} is dense in its closure, it suffices to show that

$$\text{if } (v, w)_{\mathcal{H}} = 0 \text{ for all } w = (G, f_0, f_1) \in \mathcal{R}, \text{ then } v = 0,$$

where $(\cdot, \cdot)_{\mathcal{H}}$ is the inner product on $\mathcal{H} \equiv L^2(T) \times H^{\frac{3}{2}}(\partial\Omega) \times H^0(\partial\Omega)$. Write $v = (H, v_0, v_1)$, $w = (G, f_0, f_1)$, and suppose that

$$(H, G)_{L^2(T)} + (v_0, f_0)_{\frac{3}{2}, \partial} + (v_1, f_1)_{0, \partial} = 0 \text{ for all } (G, f_0, f_1) \in \mathcal{R},$$

where $(\cdot, \cdot)_{L^2(T)}$, $(\cdot, \cdot)_{\frac{3}{2}, \partial}$, and $(\cdot, \cdot)_{0, \partial}$ are the inner products in $L^2(T)$, $H^{\frac{3}{2}}(\partial\Omega)$, and $H^0(\partial\Omega)$, respectively. By definition of \mathcal{R} and the fact that \mathcal{L} is injective, the above means

$$(H, \ddot{f} - \kappa \overline{\Delta} \partial_{\nu} f)_{L^2(T)} + (v_0, \dot{f}(0))_{\frac{3}{2}, \partial} + (v_1, \dot{f}(0))_{0, \partial} = 0 \text{ for all } f \in X_T^3(\partial\Omega). \quad (3.2)$$

Assume first that $f \in X_T^{3,+}(\partial\Omega)$. In this case (3.2) becomes

$$(H, \ddot{f} - \kappa \overline{\Delta} \partial_{\nu} f)_{L^2(T)} = 0. \quad (3.3)$$

Suppose, further, that $H \in X_T^{3,-}(\partial\Omega)$. Then

$$\begin{aligned} (H, \ddot{f} - \kappa \overline{\Delta} \partial_{\nu} f)_{L^2(T)} &= \int_{[0,T] \times \partial\Omega} H(\ddot{f} - \kappa \overline{\Delta} \partial_{\nu} f) \\ &= \int_{\partial\Omega} \int_0^T H \ddot{f} dt dx - \kappa \int_{\partial\Omega} \int_0^T H \overline{\Delta} \partial_{\nu} f dt dx \end{aligned} \quad (3.4)$$

Integrating by parts in the time variable the first term,

$$\int_{\partial\Omega} \int_0^T H \ddot{f} dt dx = \int_{\partial\Omega} \int_0^T \ddot{H} f dt dx, \quad (3.5)$$

where we used that $f \in X_T^{3,+}(\partial\Omega)$ and $H \in X_T^{3,-}(\partial\Omega)$. For the second term in (3.4), switch the order of integration and integrate by parts the Laplacian term to get

$$\int_{\partial\Omega} \int_0^T H \overline{\Delta} \partial_{\nu} f dt dx = \int_0^T \int_{\partial\Omega} \overline{\Delta} H \partial_{\nu} f dx dt.$$

Next, consider the harmonic extensions of f and H to the whole of Ω , which we still denote by f , and H , respectively. Letting $\varphi(r) = r^2$, using Green's identity, and arguing as in section 2.3,

$$\int_{\Omega} (\varphi \overline{\Delta} H \Delta f - f \Delta(\varphi \overline{\Delta} H)) = \int_{\partial\Omega} (\overline{\Delta} H \partial_{\nu} f - f \partial_{\nu}(\varphi \overline{\Delta} H)) = \int_{\partial\Omega} (\overline{\Delta} H \partial_{\nu} f - f \overline{\Delta} \partial_{\nu} H), \quad (3.6)$$

where we used (2.5) so that, on $\partial\Omega$,

$$\partial_\nu(\varphi\overline{\Delta}H) = \partial_r(r^2\overline{\Delta}H) = \partial_r\Delta_{S^2}H = \Delta_{S^2}\partial_rH = \overline{\Delta}\partial_\nu H.$$

Using also (2.4),

$$\Delta(\varphi\overline{\Delta}H) = \Delta_{S^2}\partial_r^2H + \Delta_{S^2}\left(\frac{2}{r}\partial_rH\right) + \Delta_{S^2}\left(\frac{1}{r^2}\Delta_{S^2}H\right) = \Delta_{S^2}\Delta H = 0.$$

And as $\Delta f = 0$, (3.6) gives

$$\int_{\partial\Omega} \overline{\Delta}H\partial_\nu f = \int_{\partial\Omega} f\overline{\Delta}\partial_\nu H. \quad (3.7)$$

Using (3.5) and (3.7) into (3.4) yields,

$$(H, \ddot{f} - \overline{\Delta}\partial_\nu f)_{L^2(T)} = (\ddot{H} - \overline{\Delta}\partial_\nu H, f)_{L^2(T)}.$$

Since this holds for all $f \in X_T^{3,+}(\partial\Omega)$ we conclude, from (3.2) and the density of $X_T^{3,+}(\partial\Omega)$ in $L^2(T)$, that $(\ddot{H} - \overline{\Delta}\partial_\nu H, f)_{L^2(T)} = 0$ for all $f \in L^2(T)$, and thus

$$\ddot{H} - \overline{\Delta}\partial_\nu H = 0.$$

Recall that $H(T) = \dot{H}(T) = 0$ because $H \in X_T^{3,-}(\partial\Omega)$. But solutions to the above equation, with these boundary conditions, are unique; this follows using the same argument used to show that \mathcal{L}^{-1} is injective, i.e., using energy estimates, except with the roles of 0 and T reversed. Thus $H = 0$. Since $X_T^{3,-}(\partial\Omega)$ is dense in $L^2(T)$, we conclude from the continuity of the inner product that if $(v, w)_{\mathcal{H}} = 0$ for all $w \in \mathcal{R}$, then $v = (0, v_0, v_1)$. (3.2) therefore reduces to

$$(v_0, f(0))_{\frac{3}{2}, \partial} + (v_1, \dot{f}(0))_{0, \partial} = 0 \text{ for all } f \in X_T^3(\partial\Omega).$$

But if f is an arbitrary element of $X_T^3(\partial\Omega)$, then $f(0)$ and $\dot{f}(0)$ are arbitrary elements of $H^3(\partial\Omega)$ and $H^{\frac{3}{2}}(\partial\Omega)$, respectively. Since these last two spaces are dense in, respectively, $H^{\frac{3}{2}}(\partial\Omega)$ and $H^0(\partial\Omega)$, we conclude that

$$(v_0, f_0)_{\frac{3}{2}, \partial} + (v_1, f_1)_{0, \partial} = 0 \text{ for all } (f_0, f_1) \in H^{\frac{3}{2}}(\partial\Omega) \times H^0(\partial\Omega),$$

and thus $v = 0$, as desired. \square

Proof of proposition 1.1-(iii): We already know that $\overline{\mathcal{L}^{-1}}$ is defined on the whole of \mathcal{H} , and any $u \in \mathcal{H}$ is the limit of a sequence in \mathcal{R} . Let $y = \overline{\mathcal{L}^{-1}}(u)$, and take a sequence $\{u_\ell\} \subset \mathcal{R}$ converging in \mathcal{H} to u . Then

$$\overline{\mathcal{L}^{-1}}(u_\ell) = \mathcal{L}^{-1}(u_\ell) \equiv y_\ell.$$

By construction, \mathcal{L}^{-1} is the inverse of \mathcal{L} defined on $X_T^3(\partial\Omega)$, thus $y_\ell \in X_T^3(\partial\Omega)$. As we showed that \mathcal{L}^{-1} is a continuous map (see (3.1)), we have

$$\|y - y_\ell\|_{X_T^{\frac{3}{2}}(\partial\Omega)} = \|\overline{\mathcal{L}^{-1}}(u) - \mathcal{L}^{-1}(u_\ell)\|_{X_T^{\frac{3}{2}}(\partial\Omega)} = \|\overline{\mathcal{L}^{-1}}(u - u_\ell)\|_{X_T^{\frac{3}{2}}(\partial\Omega)} \leq C \|u - u_\ell\|_{\mathcal{H}},$$

implying that $y_\ell \rightarrow y$ in $X_T^{\frac{3}{2}}(\partial\Omega)$, and thus $y \in \overline{X_T^3(\partial\Omega)}^{X_T^{\frac{3}{2}}(\partial\Omega)}$.

Injectivity also follows from the continuity of $\overline{\mathcal{L}^{-1}}$. Say $\overline{\mathcal{L}^{-1}}(u) = 0$, and let $\{u_\ell\}$ be as above. Then

$$0 = \overline{\mathcal{L}^{-1}}(u) = \overline{\mathcal{L}^{-1}}(\lim u_\ell) = \lim \overline{\mathcal{L}^{-1}}(u_\ell),$$

which implies $u_\ell = 0$ for every ℓ since $\overline{\mathcal{L}^{-1}} = \mathcal{L}^{-1}$ on \mathcal{R} and \mathcal{L}^{-1} is injective. Thus $u = 0$. \square

Proof of theorem 1.3: The existence and uniqueness of a weak solution follows at once from proposition 1.1, upon noticing that if $f \in X_T^{\frac{3}{2}}(\partial\Omega)$, then, by elliptic theory, its harmonic extension is in $X_T^2(\Omega)$. \square

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DEPARTMENT OF MATHEMATICS, VANDERBILT UNIVERSITY, NASHVILLE, TN 37240, USA
E-mail address: `marcelo.disconzi@vanderbilt.edu`